Asymptotically good binary linear codes with asymptotically good self-intersection spans

Hugues Randriambololona

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Abstract

If \( C \) is a binary linear code, let \( C^{(2)} \) be the linear code spanned by intersections of pairs of codewords of \( C \). We construct an asymptotically good family of binary linear codes such that, for \( C \) ranging in this family, the \( C^{(2)} \) also form an asymptotically good family. For this we use algebraic-geometry codes, concatenation, and a fair amount of bilinear algebra.

More precisely, the two main ingredients used in our construction are, first, a description of the symmetric square of an odd degree extension field in terms only of field operations of small degree, and second, a recent result of Garcia-Stichtenoth-Bassa-Beelen on the number of points of curves on such an odd degree extension field.

1 Statement of result

Let \( q \) be a prime power, and \( \mathbb{F}_q \) the field with \( q \) elements. For any integer \( n \geq 1 \), let \( * \) denote coordinatewise multiplication in the vector space \((\mathbb{F}_q)^n\), so

\[
(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n).
\]

For \( C \subset (\mathbb{F}_q)^n \) a linear subspace, i.e. a \( q \)-ary linear code of length \( n \), let

\[
C * C = \{ c * c' \mid c, c' \in C \}
\]

and let

\[
C^{(2)} = (C * C) = \{ \sum_{c, c' \in C} \alpha_{c, c'} c * c' \mid \alpha_{c, c'} \in \mathbb{F}_q \}
\]

be the linear span of \( C * C \). In fact the set \( C * C \) is stable under multiplication by scalars (because \( C \) is), so \( C^{(2)} \) can equivalently be defined as just the additive span of \( C * C \).

Study of the behavior of codes under this operation \( * \) is a natural problem. But motivation also comes from applications, such as the analysis of bilinear algorithms [5]. There are also links with secret-sharing and multi-party computation systems [1].
Remark that the support of $c \ast c'$ is the intersection of the supports of $c$ and $c'$. Sometimes we will call $C^{(2)}$ the self-intersection span of $C$. We will be especially interested in the case $q = 2$, where a codeword can indeed be identified with its support, unambiguously.

Write $R(C)$ and $\delta(C)$ for the rate and relative minimum distance of $C$. As a shortcut, write also $R^{(2)}(C) = R(C^{(2)})$ and $\delta^{(2)}(C) = \delta(C^{(2)})$. It is easily seen that these functions satisfy:

$$R^{(2)} \geq R \quad \delta^{(2)} \leq \delta \quad (2)$$

(see Proposition 11 below; for $q = 2$ one even has the stronger result that $C$ is a subcode of $C^{(2)}$, since then $c \ast c = c$ for all $c$).

Recall that a family of codes $C_i$ of length going to infinity is said asymptotically good if both $R(C_i)$ and $\delta(C_i)$ admit a positive asymptotic lower bound.

Theorem 1. For any prime power $q$ (e.g., $q = 2$), there exists an asymptotically good family of $q$-ary linear codes $C_i$ whose self-intersection spans $C_i^{(2)}$ also form an asymptotically good family.

Keeping (2) in mind, we can rephrase the theorem as asking for $\epsilon, \epsilon' > 0$ such that $\liminf R(C_i) \geq \epsilon$ and $\liminf \delta^{(2)}(C_i) \geq \epsilon'$. Our proof will be constructive, for example for $q = 2$ we will give an explicit construction with $\epsilon = 1/651$ and $\epsilon' = 1/1575$.

There is a certain similarity between our object of interest and the theory of linear intersecting codes [2][6]. Recall that a linear code $C$ is said intersecting if $c \ast c'$ is non-zero for all non-zero $c, c' \in C$ (and this could be refined by requiring $c \ast c'$ to have at least a certain prescribed weight). Although none of these notions imply the other, it turns out that methods used to produce intersecting codes often produce codes having a good $\delta^{(2)}$. This is the case, for example, for intersecting codes constructed as evaluation codes [8].

Suppose we are given an algebra $\mathcal{F}$ of functions, admitting a nice notion of “degree”, and which can be evaluated at a certain set of points $X$. We then define a linear code $C_D$ as the image of the space $\mathcal{F}(D)$ of functions of degree at most $D$ under this evaluation map. For example, $\mathcal{F}$ could be the algebra of polynomials in one or several indeterminates over a finite field, giving rise to Reed-Solomon or Reed-Muller codes. Or $\mathcal{F}$ could be the function field of an algebraic curve, giving rise to Goppa’s algebraic-geometry codes. In all these situations, bounds on the parameters of $C_D$ can be deduced from $D$ and the cardinality of $X$.

Now for $f, f' \in \mathcal{F}(D)$ we have $ff' \in \mathcal{F}(2D)$, which implies $c \ast c' \in C_{2D}$ for all $c, c' \in C_D$. Applying the aforementioned bounds to $C_{2D}$, we find that $C_D$ is intersecting provided $D$ is suitably chosen. But in fact, by linearity, the argument just above gives the stronger result $C^{(2)}_D \subset C_{2D}$, from which a lower bound on $\delta^{(2)}(C_D) \geq \delta(C_{2D})$ can be deduced in the same way. Remark that to have a lower bound on $R(C_D)$ requires in general $D$ to be large, while a lower bound on $\delta(C_{2D})$ requires $2D$ to be small with respect to the cardinality of $X$.
Unfortunately, with the present techniques, if the size $q$ of the field is too small, these two requirements become contradictory when one lets the length of the code go to infinity. If one is interested only in constructing intersecting codes, a solution is to work first over an extension field, and then conclude with a concatenation argument [7]; this is because a concatenation of intersecting codes is intersecting. But in the problem we study, things do not behave so nicely. So one could say that our goal in the present work is to find a concatenation procedure that is compatible with the intersection span operation. For this we will put real care in constructing both the inner and the outer codes.

2 Bilinear study of field extensions

Let $V$ be a vector space of dimension $r$ over $\mathbb{F}_q$, and let $V^\vee$ be its dual vector space. Let also Sym($V; \mathbb{F}_q$) be the space of symmetric bilinear forms on $V$. If $\lambda \in V^\vee$ is a linear form on $V$, we can define

$$\lambda \otimes^2 : V \times V \rightarrow \mathbb{F}_q \quad (v, w) \mapsto \lambda(v)\lambda(w)$$

which is a symmetric bilinear form on $V$.

Lemma 2. Let $\lambda_1, \ldots, \lambda_r$ be a basis of $V^\vee$. Then the $\frac{(r+1)(r+2)}{2}$ elements $\lambda_i \otimes^2$ for $1 \leq i \leq r$ and $(\lambda_i + \lambda_j) \otimes^2$ for $1 \leq i < j \leq r$ form a basis of Sym($V; \mathbb{F}_q$).

Proof. Using $\lambda_1, \ldots, \lambda_r$ as coordinate functions we can suppose $V = (\mathbb{F}_q)^r$. Then $\lambda_i \otimes^2$ is the symmetric bilinear form

$$(v, w) \mapsto v_i w_i$$

(3)

and $(\lambda_i + \lambda_j) \otimes^2$ is $(v, w) \mapsto (v_i + v_j)(w_i + w_j)$, hence $(\lambda_i + \lambda_j) \otimes^2 - \lambda_i \otimes^2 - \lambda_j \otimes^2$ is

$$(v, w) \mapsto v_i w_j + v_j w_i.$$ 

(4)

Then we conclude by recognizing these (3) and (4) as forming the standard basis of Sym(($\mathbb{F}_q)^r; \mathbb{F}_q$).

We will now be interested in the case $V = \mathbb{F}_{q^r}$ is an extension field, which can indeed be considered as a vector space over $\mathbb{F}_q$, and we let $\gamma_1, \ldots, \gamma_r$ be a basis (for example $\gamma_i = \gamma^{i-1}$ for some choice of a primitive element $\gamma \in \mathbb{F}_{q^r}$). Let also $\text{Tr} : \mathbb{F}_{q^r} \rightarrow \mathbb{F}_q$ denote the trace function. To each $a \in \mathbb{F}_{q^r}$ we can associate a linear form

$$t_a : \mathbb{F}_{q^r} \rightarrow \mathbb{F}_q \quad x \mapsto \text{Tr}(ax).$$

The following is well known:
Lemma 3. The map
\[ F_{q^r} \rightarrow (\mathbb{F}_{q^r})^\vee \]
\[ a \mapsto t_a \]
is an isomorphism of $\mathbb{F}_q$-vector spaces. In particular, $t_{\gamma_1}, \ldots, t_{\gamma_r}$ form a basis of $(\mathbb{F}_{q^r})^\vee$.

As a field, $\mathbb{F}_{q^r}$ is endowed with its usual multiplication law, which we will denote by $m_0$, so
\[ m_0(x, y) = xy \]
for $x, y \in \mathbb{F}_{q^r}$. For any integer $j \geq 1$, we can also define a “twisted multiplication law” $m_j$ by
\[ m_j(x, y) = xy^{q^j} + x^j y. \]
Remark that these maps are symmetric and $\mathbb{F}_q$-bilinear (although not $\mathbb{F}_{q^r}$-bilinear in general).

Proposition 4. Choose an ordering of the set \( \{t_{\gamma_1}\}_{1 \leq i \leq r} \cup \{t_{\gamma_i + \gamma_j}\}_{1 \leq i < j \leq r} \) and rename its elements accordingly, say:
\[ \{t_{\gamma_i}\}_{1 \leq i \leq r} \cup \{t_{\gamma_i + \gamma_j}\}_{1 \leq i < j \leq r} = \{\phi_1, \ldots, \phi_{r(r+1)/2}\}. \]
Then:
- The family \( \{\phi_{\otimes 2} \otimes 2^i\}_{1 \leq i \leq r} \) is a basis of $\text{Sym}(\mathbb{F}_{q^r}; \mathbb{F}_q)$.
- If $r = 2s + 1$ is odd, the family \( \{t_{\gamma_i} \circ m_j\}_{1 \leq i \leq r, 0 \leq j \leq s} \) is a basis of $\text{Sym}(\mathbb{F}_{q^r}; \mathbb{F}_q)$.

Proof. The first claim is a consequence of Lemma 2 and Lemma 3. To prove the second claim, start by remarking that the given family has the correct size $r(s+1) = \frac{r(r+1)}{2}$. It suffices thus to show that it is a generating family, and for this (because of the first claim) it suffices to show that each $t_{a_{\otimes 2}}$, for $a \in \mathbb{F}_{q^r}$, can be written as a linear combination of the $t_0 \circ m_j$, for $b \in \mathbb{F}_{q^r}$ and $0 \leq j \leq s$. However for any $x, y \in \mathbb{F}_{q^r}$ we have
\[
\text{Tr}(ax) \text{Tr}(ay) = (ax + a^q x^q + \cdots + a^{q^s} x^{q^s})(ay + a^{q^j} y^q + \cdots + a^{q^{s+1}} y^{q^{s+1}})
= \text{Tr}(a^2 xy) + \sum_{1 \leq j \leq s} \text{Tr}(a^{1+q^j} (x y^{q^j} + x^{q^j} y))
\]
which can be restated
\[ t_{a_{\otimes 2}} = \sum_{0 \leq j \leq s} t_{a_{1+q^j}} \circ m_j \]
as wanted. \qed
From now on we suppose $r = 2s + 1$ is odd, so $\frac{r(r+1)}{2} = (s+1)(2s+1)$. Consider the symmetric $\mathbb{F}_q$-bilinear maps
\[
\Phi = (\phi_1^{s+2}, \ldots, \phi_{(s+1)(2s+1)}^{s+2}) : \mathbb{F}_{q^{2s+1}} \times \mathbb{F}_{q^{2s+1}} \longrightarrow (\mathbb{F}_q)^{(s+1)(2s+1)}
\]
and
\[
\Psi = (m_0, \ldots, m_s) : \mathbb{F}_{q^{2s+1}} \times \mathbb{F}_{q^{2s+1}} \longrightarrow (\mathbb{F}_q^{2s+1})^{s+1}.
\]
Proposition 4 can then be restated as follows:

**Corollary 5.** There is an isomorphism of $\mathbb{F}_q$-vector spaces
\[
\theta : (\mathbb{F}_q)^{(s+1)(2s+1)} \rightarrow (\mathbb{F}_q^{2s+1})^{s+1}
\]
such that
\[
\theta \circ \Phi = \Psi.
\]

**Proof.** Set $r = 2s + 1$, use the $t_{\gamma}$ as coordinate functions on $\mathbb{F}_q^{r}$ as allowed by Lemma 3, and define $\theta$ as the invertible linear transformation that maps the first basis of $\text{Sym}(\mathbb{F}_q^{r}; \mathbb{F}_q)$ given in Proposition 4 to the second one. \(\square\)

**Remark 6.** For the more sophisticated reader, recall that the symmetric square of a vector space $V$ over $\mathbb{F}_q$ can be defined, for our purpose, as the dual of the space of symmetric bilinear forms on it: $S^2\mathbb{F}_q V = \text{Sym}(V; \mathbb{F}_q)^\vee$. We let $(v, w) \mapsto v \cdot w$ be the universal symmetric bilinear map $V \times V \rightarrow S^2\mathbb{F}_q V$, where $v \cdot w \in S^2\mathbb{F}_q V$ is the “evaluation” element that sends $F \in \text{Sym}(V; \mathbb{F}_q)$ to $F(v, w)$. Recall also the universal property of the symmetric square: for any $\mathbb{F}_q$-vector space $W$, there is a natural identification
\[
\{ \text{symmetric bilinear maps} \} = \{ \text{linear maps} \}
\]
as $\mathbb{F}_q$-vector spaces, where a linear map $f : S^2\mathbb{F}_q V \rightarrow W$ corresponds to the symmetric bilinear map $(v, w) \mapsto f(v \cdot w)$.

So, in the case $V = \mathbb{F}_{q^{2s+1}}$, the symmetric bilinear maps $\Phi$ and $\Psi$ give rise to linear maps $\overline{\Phi}$ and $\overline{\Psi}$ on $S^2\mathbb{F}_q \mathbb{F}_{q^{2s+1}}$, and Proposition 4 expresses that these
\[
\overline{\Phi} : S^2\mathbb{F}_q \mathbb{F}_{q^{2s+1}} \rightarrow (\mathbb{F}_q)^{(s+1)(2s+1)}
\]
and
\[
\overline{\Psi} : S^2\mathbb{F}_q \mathbb{F}_{q^{2s+1}} \rightarrow (\mathbb{F}_q^{2s+1})^{s+1}
\]
are isomorphisms of $\mathbb{F}_q$-vector spaces (while $\theta = \overline{\Psi} \circ \overline{\Phi}^{-1}$ in Corollary 5).

A similar result can be given in the case of an even degree extension $\mathbb{F}_{q^2}$, with only one minor change. Indeed, in this case remark that one has $(xy^{q^t} + x^{q^t}y^{q^t})^{q^t} = x^{q^t}y + xy^{q^t}$ for all $x, y \in \mathbb{F}_{q^{2s}}$, which means that $m_s$ takes values in
the subfield $\mathbb{F}_{q^s}$ of $\mathbb{F}_{q^2}$. Then the very same arguments as before show that
$m_0, \ldots, m_s$ induce an isomorphism of $\mathbb{F}_q$-vector spaces

$$S_q^2 \mathbb{F}_{q^2} \sim \to (\mathbb{F}_{q^2})^s \times \mathbb{F}_q,$$

and composing with traces gives a basis of $\text{Sym}(\mathbb{F}_{q^2}; \mathbb{F}_q)$ in this case also.

### 3 Bilinear study of concatenated codes

If $\mathcal{A}$ is a vector space of finite dimension over $\mathbb{F}_q$, if $n \geq 1$ is an integer and $C \subset \mathcal{A}^n$ is a linear subspace, and if $f : \mathcal{A} \to \mathcal{B}$ is a linear map from $\mathcal{A}$ to another vector space $\mathcal{B}$, we denote by $f(C) \subset \mathcal{B}^n$ the subspace obtained by applying $f$ componentwise to the "codewords" of $C$:

$$f(C) = \{(f(c_1), \ldots, f(c_n)) \in \mathcal{B}^n \mid c = (c_1, \ldots, c_n) \in C \subset \mathcal{A}^n\}.$$

Also if $C' \subset \mathcal{A}^n$ is a code of the same length over another linear alphabet $\mathcal{A}'$, and if $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{B}$ is a bilinear map, we denote by $(F(C, C')) \subset \mathcal{B}^n$ the linear span of the set of elements obtained by applying $F$ componentwise to pairs of codewords in $C$ and $C'$:

$$(F(C, C')) = \left\{\sum_{c \in C, c' \in C'} \alpha_{c, c'}(F(c_1, c'_1), \ldots, F(c_n, c'_n)) \in \mathbb{F}_q \mid \alpha_{c, c'} \in \mathbb{F}_q^s\right\} \quad (5)$$

which generalizes (1).

We will be interested in the case $\mathcal{A} = \mathbb{F}_{q^{2s+1}}$ is an odd degree extension field of $\mathbb{F}_q$. Recall the notations from the previous section. First we have the linear map

$$\phi = (\phi_1, \ldots, \phi_{(s+1)(2s+1)}) : \mathbb{F}_{q^{2s+1}} \to (\mathbb{F}_q)^{(s+1)(2s+1)}$$

as well as the symmetric bilinear map

$$\Phi = (\phi_1^{\otimes 2}, \ldots, \phi_{(s+1)(2s+1)}^{\otimes 2}) : \mathbb{F}_{q^{2s+1}} \times \mathbb{F}_{q^{2s+1}} \to (\mathbb{F}_q)^{(s+1)(2s+1)}.$$  

If $C \subset (\mathbb{F}_{q^{2s+1}})^n$ is a linear code of length $n$ over $\mathbb{F}_{q^{2s+1}}$, we will consider $\phi(C)$ and $(\Phi(C, C))$ as codes of length $N = (s+1)(2s+1)n$ over $\mathbb{F}_q$, using the natural identification $((\mathbb{F}_q)^{(s+1)(2s+1)})^n = (\mathbb{F}_q)^N$. Then:

**Lemma 7.** With these notations,

$$(\Phi(C, C)) = \phi(C)^{(2)}.$$

**Proof.** Direct consequence of the definitions. \(\square\)

We also have the symmetric $\mathbb{F}_q$-bilinear maps

$$m_j : \mathbb{F}_{q^{2s+1}} \times \mathbb{F}_{q^{2s+1}} \to \mathbb{F}_{q^{2s+1}}$$

for $0 \leq j \leq s$, from which we formed

$$\Psi = (m_0, \ldots, m_s) : \mathbb{F}_{q^{2s+1}} \times \mathbb{F}_{q^{2s+1}} \to (\mathbb{F}_{q^{2s+1}})^{s+1}.$$
Remark that \( m_0 \) is not only \( \mathbb{F}_q \)-bilinear, it is also \( \mathbb{F}_{q^{2s+1}} \)-bilinear. So if the code \( C \subset (\mathbb{F}_{q^{2s+1}})^n \) is \( \mathbb{F}_{q^{2s+1}} \)-linear, then so is \( \langle m_0(C,C) \rangle \). In fact \( \langle m_0(C,C) \rangle = C^{(2)} \) provided now componentwise multiplication \( \ast \) is meant over \( \mathbb{F}_{q^{2s+1}} \).

On the other hand, for \( j \geq 1 \), \( m_j \) is only \( \mathbb{F}_q \)-bilinear. So \( \langle m_j(C,C) \rangle \) will only be a \( \mathbb{F}_q \)-linear subspace of \( (\mathbb{F}_{q^{2s+1}})^n \) (and similarly for \( \langle \Psi(C,C) \rangle \)). Nevertheless we will still define the weight of a codeword in \( \langle m_j(C,C) \rangle \) and its minimal distance \( d_{\min}(\langle m_j(C,C) \rangle) \) as the usual weight and distance taken in \( (\mathbb{F}_{q^{2s+1}})^n \), that is, over the alphabet \( \mathbb{F}_{q^{2s+1}} \).

**Proposition 8.** With the notations above,

\[
d_{\min}(\phi(C)^{(2)}) \geq \min_{0 \leq j \leq s} d_{\min}(\langle m_j(C,C) \rangle).
\]

*Proof.* Let \( c \in \phi(C)^{(2)} \) be a codeword. We have to show that if \( c \) has weight

\[
w < d_{\min}(\langle m_j(C,C) \rangle)
\]

for all \( 0 \leq j \leq s \), then it is the zero codeword.

Here \( c \) is seen as a word of length \( n \) over the alphabet \( \mathbb{F}_q \), but we can also see it as a word of length \( n \) over the alphabet \( (\mathbb{F}_q)^{(s+1)(2s+1)} \), and as such obviously it has weight

\[
\hat{w} \leq w.
\]

Now, using Corollary 5 and Lemma 7, we apply \( \theta \) blockwise to get a codeword \( \theta(c) \in \langle \Psi(C,C) \rangle \). Since \( \theta \) is invertible, we see that, considered as a word of length \( n \) over the alphabet \( (\mathbb{F}_{q^{2s+1}})^{s+1} \), this \( \theta(c) \) has the same weight \( \hat{w} \).

If we denote by \( \pi_0, \ldots, \pi_s \) the \( s+1 \) coordinate projections \( (\mathbb{F}_{q^{2s+1}})^{s+1} \twoheadrightarrow \mathbb{F}_{q^{2s+1}} \), then by construction we have \( m_j = \pi_j \circ \Psi \), so applying \( \pi_j \) blockwise we get a codeword \( \pi_j(\theta(c)) \in \langle m_j(C,C) \rangle \), of weight at most \( \hat{w} \). But then, \( \pi_j(\theta(c)) \) is the zero codeword because of (6) and (7), and since this holds for all \( j \), we conclude that \( \theta(c) \) is zero, hence \( c \) is zero. \( \square \)

**Remark 9.** This is a continuation of Remark 6. Recall from the symmetric square construction that we have a universal product \( \cdot : \mathbb{F}_{q^{2s+1}} \times \mathbb{F}_{q^{2s+1}} \twoheadrightarrow S_2^{(2)} \mathbb{F}_{q^{2s+1}} \). The underlying notion in the proof of Proposition 8 is then that of the “universal symmetric bilinear span”

\[
\langle C \cdot C \rangle \subset (S_2^{(2)} \mathbb{F}_{q^{2s+1}})^n
\]

constructed from \( C \) and \( \cdot \) as in (5), and of which \( \langle \Phi(C,C) \rangle = \phi(C)^{(2)} \) and \( \langle \Psi(C,C) \rangle \) are two incarnations, under the invertible linear changes of alphabets \( \Phi \) and \( \Psi \). In particular the weight \( \hat{w} \) in (7) should be interpreted as the weight of \( c \) considered as a word over the alphabet \( S_2^{(2)} \mathbb{F}_{q^{2s+1}} \).

Now let \( K \) be a finite field (we will apply both cases \( K = \mathbb{F}_q \) and \( K = \mathbb{F}_{q^{2s+1}} \)), and let \( \ast \) denote coordinatewise multiplication in the vector space \( K^n \), which is
a symmetric $K$-bilinear map $K^n \times K^n \rightarrow K^n$. If $C, C' \subset K^n$ are two linear codes of the same length, we can define their intersection span

\[
\langle C \ast C' \rangle \subset K^n
\]

as in (5), and iteratively, setting $C^{\langle 0 \rangle}$ as the $[n, 1, n]$ repetition code, we can define higher self-intersection spans (or “powers”)

\[
C^{(t+1)} = \langle C^{(t)} \ast C \rangle
\]

for $t \geq 0$. Remark then $C^{(1)} = C$, and $C^{(2)}$ is the same as in (1). More generally, $C^{(t)}$ is the linear span of the set of coordinatewise products of $t$-uples of codewords from $C$.

**Lemma 10.** Let $t \geq 1$. If $c \in C^{(t)}$ is a codeword and if $i$ is a coordinate at which $c$ is non-zero, then there is already some $c' \in C$ that is non-zero at $i$.

**Proof.** Obvious.

Now given a code $C \subset K^n$, for each integer $t \geq 0$, we can define the “higher” dimension $\dim^{(t)}$, distance $d^{(t)}_{\min}$, rate $R^{(t)}$, and relative distance $\delta^{(t)}$ of $C$, as those parameters for $C^{(t)}$.

**Proposition 11.** For all $n \geq 1$ and for all (non-zero) code $C \subset K^n$, we have, for all $t \geq 0$,

\[
\dim^{(t+1)}(C) \geq \dim^{(t)}(C)
\]

and

\[
d^{(t+1)}_{\min}(C) \leq d^{(t)}_{\min}(C).
\]

**Proof.** For $t = 0$ these inequalities hold by convention, so we suppose $t \geq 1$. Let $k_t = \dim^{(t)}(C)$, and let $S \subset \{1, \ldots, n\}$ be an information set of coordinates for $C^{(t)}$. Without loss of generality we can suppose $S = \{1, \ldots, k_t\}$. Let $G_t$ be the generating matrix of $C^{(t)}$ put in systematic form with respect to $S$. If $c$ is the $i$-th line of $G_t$, then $c \in C^{(t)}$ has a 1 at coordinate $i$ and is zero over $S \setminus \{i\}$. By Lemma 10 we can find $c' \in C$ that is non-zero at $i$, hence $c \ast c' \in C^{(t+1)}$ is non-zero at $i$ and zero over $S \setminus \{i\}$. Letting $i$ vary we see that $C^{(t+1)}$ has full rank over $S$, hence $\dim C^{(t+1)} \geq k_t$. This is the first inequality.

Now let $d_t = d^{(t)}_{\min}(C)$ and let $c \in C^{(t)}$ be a codeword of weight $d_t$. Let $i$ be a non-zero coordinate of $c$, so by Lemma 10 we can find $c' \in C$ that is non-zero at $i$. Then $c \ast c' \in C^{(t+1)}$ is non-zero at $i$, so it is not the zero codeword, and its support is a subset of the support of $c$, hence $d^{(t+1)}_{\min}(C) \leq d_t$. This is the second inequality.

**Corollary 12.** Let $n \geq k \geq 1$ and $s \geq 0$ be integers, and let $N = (s+1)(2s+1)n$. Let also $\phi : \mathbb{F}_q^{2s+1} \rightarrow (\mathbb{F}_q)^{(2s+1)(2s+1)}$ be the $\mathbb{F}_q$-linear map defined earlier. Then, for any $\mathbb{F}_q^{2s+1}$-linear $[n, k]$ code $C$, the “concatenated” code $\phi(C)$ is a $\mathbb{F}_q$-linear $[N, (2s+1)k]$ code, and more precisely we have:

(i) $\dim \phi(C) = (2s + 1) \dim C$
\( d_{\min}(\phi(C)) \geq d_{\min}^{(1+q^r)}(C) \)

\( R(\phi(C)) = \frac{1}{s+1} R(C) \)

\( \delta^{(2)}(\phi(C)) \geq \frac{1}{(s+1)(2s+1)} \delta^{(1+q^r)}(C) \)

where, in the left, parameters (and coordinatewise product) are meant over \( \mathbb{F}_q \), and in the right, they are over \( \mathbb{F}_{q^{2s+1}} \).

**Proof.** Remark that \( \phi \) is injective, since it was constructed by extending a basis \( t_{\gamma_1}, \ldots, t_{\gamma_r} \) of \( (\mathbb{F}_q^r)^{\vee} \), where \( r = 2s + 1 \). This implies that \( \phi(C) \) has dimension \( rk \), from which (i) and (iii) follow.

On the other hand, since \( m_0(x,y) = xy \) and \( m_j(x,y) = xy^{q^j} + x^{q^j}y \) for \( j \geq 1 \), we find

\[ \langle m_j(C,C) \rangle \subset C^{(1+q^r)} \]

for all \( j \geq 0 \). In this inclusion, the right-hand side is a \( \mathbb{F}_{q^{2s+1}} \)-linear code, while in general the left-hand side is only a \( \mathbb{F}_q \)-linear subspace. Nevertheless this implies

\[ d_{\min}(\langle m_j(C,C) \rangle) \geq d_{\min}^{(1+q^r)}(C) \]

and together with Propositions 8 and 11, this gives (ii), and then (iv). \( \square \)

## 4 Algebraic-geometry codes

Let \( K \) be a finite field. If \( X \) is a (smooth, projective, absolutely irreducible) curve over \( K \), we define a divisor \( D \) on \( X \) as a formal sum of (closed) points of \( X \), to which one associates the \( K \)-vector space \( L(D) \), of dimension \( l(D) \), made of the functions \( f \) on \( X \) having poles at most \( D \). If \( D \) and \( G \) have disjoint support, we can define an evaluation map

\[ ev_{D,G} : \quad L(D) \longrightarrow K^n \]

\[ f \quad \mapsto \quad (f(P_1), \ldots, f(P_n)) \]

and an evaluation code

\[ C(D,G) \subset K^n \]

as the image of this \( K \)-linear map \( ev_{D,G} \). Then, from the preceding properties of \( l(D) \) we deduce:
Lemma 13 (Goppa). Suppose $g \leq \text{deg}(D) < n$. Then

$$\dim C(D, G) = l(D) \geq \text{deg}(D) + 1 - g$$

and

$$d_{\min}(C(D, G)) \geq n - \text{deg}(D).$$

Evaluation codes also behave well with regard to our intersection span operations:

Lemma 14. For any integer $t \geq 0$ we have

$$C(D, G)^{(t)} \subset C(tD, G).$$

Proof. This is true for $t = 0$, so by induction it suffices to show $\langle C(tD, G) \ast C(D, G) \rangle \subset C((t + 1)D, G)$, or more generally,

$$\langle C(D, G) \ast C(D', G) \rangle \subset C(D + D', G)$$

for any divisors $D, D'$ with supports disjoint from $G$. But for $c \in C(D, G)$ and $c' \in C(D', G)$, write $c = \text{ev}(f)$ and $c' = \text{ev}(f')$ with $f \in L(D)$ and $f' \in L(D')$, and then $c \ast c' = \text{ev}(ff')$ with $ff' \in L(D + D')$, from which the conclusion follows.

Proposition 15. Let $q$ be a prime power, and $s \geq 0$ an integer. Let $X$ be a curve over $\mathbb{F}_{q^s+1}$, of genus $g$, and suppose that $X$ admits a set $\{P_1, \ldots, P_n\}$ of degree 1 points of cardinality

$$n > (1 + q^s)g.$$  

Let then $G = P_1 + \cdots + P_n$. Let also $D$ be a divisor on $X$ of support disjoint from $G$ and whose degree $\text{deg}(D) = m$ satisfies

$$g \leq m < (1 + q^s)^{-1}n.$$  

Finally let

$$\phi: \mathbb{F}_{q^{2s+1}} \longrightarrow (\mathbb{F}_q)^{(s+1)(2s+1)}$$

as in the previous section. Then the corresponding concatenated code

$$C = \phi(C(D, G)) \subset (\mathbb{F}_q)^{(s+1)(2s+1)n}$$

has parameters satisfying:

(i) $\dim C \geq (2s + 1)(m + 1 - g)$

(ii) $d_{\min}^{(2)}(C) \geq n - (1 + q^s)m$

(iii) $R(C) \geq \frac{1}{s + 1} \frac{m + 1 - g}{n}$
\[(iv) \quad \delta^{(2)}(C) \geq \frac{1}{(s + 1)(2s + 1)} \left(1 - \frac{(1 + q^s)m}{n}\right)\]

**Proof.** Inequalities (i) and (iii) follow from (i) and (iii) in Corollary 12 joint with Lemma 13.

Inequalities (ii) and (iv) follow from (ii) and (iv) in Corollary 12 joint with Lemma 13 applied to \(C((1+q^s)D,G)\) and Lemma 14 applied with \(t = 1+q^s\). □

For any prime power \(q\), let \(N_q(g)\) be the maximal possible number of degree 1 points of a curve of genus \(g\) over \(\mathbb{F}_q\), and let

\[
A(q) = \limsup_{g \to \infty} \frac{N_q(g)}{g}.
\]

We will now make use of a recent result of Garcia-Stichtenoth-Bassa-Beelen [3], in the following form:

**Lemma 16.** For any prime power \(q\), there exists an integer 

\[
A(q^{2s+1}) > 1 + q^s
\]

(and in fact this holds as soon as \(s\) is large enough).

**Proof.** If \(q\) is a square, one knows from [4] that \(A(q^{2s+1}) \geq (q^{2s+1})^{1/2} - 1 > 1 + q^s\) as soon as \(s\) is large enough. So suppose \(q\) is not a square, say \(q = p^{2t+1}\) with \(p\) prime. Then Theorem 1.1 of [3] gives

\[
A(q^{2s+1}) = A(p^{4st+2s+2t+1}) \geq \frac{2(p^{2st+s+t+1} - 1)}{p + 1 + \varepsilon_s}
\]

with \(\varepsilon_s \to 0\) as \(s \to \infty\), so, for \(s\) large enough, this is greater than

\[
1 + q^s = 1 + p^{2st+s}
\]

as claimed. □

From this we can finally prove our main theorem.

**Theorem 17.** Let \(q\) be a prime power, and let \(s\) be as given by Lemma 16. Then, for any real number \(\mu\) with

\[
1 < \mu < \frac{A(q^{2s+1})}{1 + q^s}
\]

there exists a family of linear codes \(C_i\) over \(\mathbb{F}_q\), of length going to infinity, satisfying

\[
\liminf_{i \to \infty} R(C_i) \geq \frac{1}{s + 1} \left(\mu - 1\right)
\]

and

\[
\liminf_{i \to \infty} \delta^{(2)}(C_i) \geq \frac{1}{(s + 1)(2s + 1)} \left(1 - \frac{(1 + q^s)\mu}{A(q^{2s+1})}\right).
\]
Proof. For any curve $X$ over $\mathbb{F}_{q^{2s+1}}$, denote by $N(X)$ the number of its degree 1 points. Let $X_1$ be a sequence of curves of genus $g_i$ going to infinity, and such that $\lim_{i} \frac{N(X_i)}{g_i} = A(q^{2s+1})$. Also choose a sequence of integers $m_i$ such that $\lim_{i} \frac{m_i}{g_i} = \mu$.

Now, given $i$ large enough, write $n_i = N(X_i) - 1$, let $P_i,0,P_i,1,\ldots,P_i,n_i$ be the degree 1 points of $X_i$, and let $D_i = m_i P_{i,0}$. Then Proposition 15 gives a code $C_i$ over $\mathbb{F}_q$ of length $(s+1)(2s+1)n_i$ with $R(C_i) \geq \frac{1}{s+1} \frac{m_i+1}{n_i}$ and $\delta(2)(C_i) \geq \left(1 - \frac{1+q^2}{n_i}\right)$, and the conclusion follows. \hfill \Box

Remark that the proof of Theorem 17 is constructive, and works also for a possibly non-optimal sequence of curves over $\mathbb{F}_{q^{2s+1}}$, by which we mean, curves satisfying $\liminf_{i} \frac{N(X_i)}{g_i} \geq A'$ for a certain $A' \leq A(q^{2s+1})$, provided still $A' > 1 + q^s$ and one replaces all occurrences of $A(q^{2s+1})$ in the theorem with $A'$.

For example [3] gives an explicit sequence of curves over $\mathbb{F}_{29}$ with $\lim_{i} \frac{N(X_i)}{g_i} \geq A' = 465/23 \approx 20.217 > 17 = 1 + 2^4$. Choosing $\mu = 186/161$ then gives an explicit sequence of binary linear codes $C_i$ of length going to infinity with $\liminf_{i} R(C_i) \geq 1/651$ and $\liminf_{i} \delta(2)(C_i) \geq 1/1575$. Of course these are only lower bounds, and it could well be that these codes actually have much better parameters.

5 Concluding remarks and open problems

Keeping Proposition 11 in mind, perhaps the most general question one can ask about the parameters of successive powers of codes is the following: given a prime power $q$, an integer $n$, and two sequences $k_1 \leq k_2 \leq k_3 \leq \ldots$ and $d_1 \geq d_2 \geq d_3 \geq \ldots$, does there exist a linear code $C \subset (\mathbb{F}_q)^n$ with $\dim^{(t)}(C) = k_t$ and $\dim^{(t)}(\min(C)) = d_t$ for all $t$? In fact, already of interest is the study of the function

$$a_{q,t}(n,d) = \max\{k \geq 0 \mid \exists C \subset (\mathbb{F}_q)^n, \dim(C) = k, \dim^{(t)}(\min(C)) \geq d\}.$$  

Proposition 11 gives $a_{q,t}(n,d) \geq a_{q,t+1}(n,d)$, and Corollary 12 gives

$$a_{q,2}(s+1)(2s+1)n,d) \geq (2s+1)a_{q^{2s+1},1+q^s}(n,d)$$

for all $s \geq 0$.

But besides parameters, one can ask for other characterizations of codes that are powers. Consider for example the “square root” problem: given a linear code $C \subset (\mathbb{F}_q)^n$, can one decide if there exists a code $C_0$ such that $C = C_0^{(2)}$, and if so, how many are there? can one construct one, or all of them, effectively?

An obvious counting argument shows that, on average, a code taken randomly in the set of all codes of given length admits one square root. However the actual distribution of squares within the set of codes of given parameters might be quite inhomogeneous, and would be interesting to study. For example, all binary codes of length 3, except two of them, are their own unique square
root. The two exceptions are: the \([3,2,2]\) parity code is not a square; the trivial \([3,3,1]\) code admits two square roots, namely itself and the \([3,2,2]\) code.

Now we turn to asymptotic properties. Define

\[
\alpha_{q,t}(\delta) = \limsup_{n \to \infty} \frac{a_{q,t}(n,\lfloor \delta n \rfloor)}{n},
\]

\[
\delta_q(t) = \sup\{\delta \geq 0 \mid \alpha_{q,t}(\delta) > 0\},
\]

and

\[
\tau(q) = \sup\{t \geq 1 \mid \delta_q(t) > 0\}.
\]

That is, \(\tau(q)\) is the supremum value (possibly \(+\infty\)) of \(t\) such that there exists an asymptotically good family of linear codes \(C_i\) over \(F_q\) whose \(t\)-th powers \(C_i^{(t)}\) also form an asymptotically good family.

From Corollary 12 one finds

\[
\alpha_{q,2}(\delta) \geq \frac{1}{s+1} \alpha_{q^{2s+1},1+q^s}(s+1)(2s+1)\delta
\]

and

\[
\delta_q(2) \geq \frac{1}{(s+1)(2s+1)} \delta_{q^{2s+1}}(1+q^s)
\]

for all \(s \geq 0\).

On the other hand, from Lemma 13 and Lemma 14 one easily finds

\[
\alpha_{q,t}(\delta) \geq \frac{1-\delta}{t} - \frac{1}{A(q)}
\]

and

\[
\delta_q(t) \geq 1 - \frac{t}{A(q)}
\]

hence

\[
\tau(q) \geq \lceil A(q) \rceil - 1
\]

(which is non-trivial only for \(q\) large).

Combining these bounds, or equivalently, eliminating \(\mu\) from the two estimates in Theorem 17, one gets

\[
\alpha_{q,2}(\delta) \geq \frac{1}{s+1} \left( \frac{1}{1+q^s} - \frac{1}{A(q^{2s+1})} \right) - \frac{2s+1}{1+q^s} \delta
\]

and

\[
\delta_q(2) \geq \frac{1}{(s+1)(2s+1)} \left( 1 - \frac{1+q^s}{A(q^{2s+1})} \right)
\]

for all \(s \geq 0\), and hence, by Lemma 16,

\[
\tau(q) \geq 2
\]
for all $q$.

When $q = p$ is prime, these estimates can be made more precise using the bound \( \frac{1}{A(p^{s+1})} \leq \frac{1}{2} \left( \frac{1}{p^s - 1} - \frac{1}{p^{s+1} - 1} \right) \) from [3]. For $p = 2$, the best choice is $s = 4$, which gives

\[
\alpha_{2,2}(\delta) \geq \frac{74}{39525} - \frac{9}{17} \delta \approx 0.001872 - 0.5294 \delta
\]

and

\[
\delta_2(2) \geq \frac{74}{20925} \approx 0.003536.
\]

This can be viewed as a more precise version of the claim $\tau(2) \geq 2$ made in the Abstract of this article. However, in the other direction, the author doesn’t know any upper bound on the $\tau(q)$, for instance, he doesn’t even know whether $\tau(2)$ is finite.

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References


